

Lipschitz perturbations of differentiable implicit functions

Oleg Makarenkov (Voronezh State University, Russia)

omakarenkov@math.vsu.ru

Abstract. Let $y = f(x)$ be a continuously differentiable implicit function solving the equation $F(x, y) = 0$ with continuously differentiable F . In this paper we show that if F_ε is a Lipschitz function such that the Lipschitz constant of $F_\varepsilon - F$ goes to 0 as $\varepsilon \rightarrow 0$ then the equation $F_\varepsilon(x, y) = 0$ has a Lipschitz solution $y = f_\varepsilon(x)$ such that the Lipschitz constant of $f_\varepsilon - f$ goes to 0 as $\varepsilon \rightarrow 0$ either. As an application we evaluate the length of time intervals where the right hand parts of some nonautonomous discontinuous systems of ODEs are continuously differentiable with respect to state variables. The latter is done as a preparatory step toward generalizing the second Bogolyubov's theorem for discontinuous systems.

1. Classical implicit function theorem. The classical implicit function theorem can be summarized as follows (see e.g. [4], Ch. X, §2, Theorems 1 and 2).

Theorem. Let X, Y, Z be Banach spaces, $x_0 \in X, y_0 \in Y$ and $r > 0$. Assume that $F : B_r(x_0) \times B_r(y_0) \rightarrow Z$ satisfies the following conditions

1. $F(x_0, y_0) = 0$,
2. F is continuous in $B_r(x_0) \times B_r(y_0)$,
3. F is continuously differentiable in $B_r(x_0) \times B_r(y_0)$ and F' has in $B_r(x_0) \times B_r(y_0)$ a bounded inverse.

Then there exists $\alpha > 0$ and $\beta > 0$ such that for any $x \in B_\alpha(x_0)$ the equation

$$F(x, y) = 0 \tag{1}$$

has a unique solution $y = f(x)$ in $B_\beta(y_0)$. Moreover, f is differentiable in $B_\alpha(x_0)$ and

$$f'(x) = - [F'_y(x, f(x))]^{-1} F'_x(x, f(x))$$

for any $x \in B_\alpha(x_0)$.

2. Main result. To prove our main result (theorem 2) we first state the following theorem 1 on the existence of the implicit function. In the case when the function F is Lipschitz theorem 1 can be derived from Clark's implicit function theorem [3], but we put the proof since it appears to be much simpler in our particular situation than that in [3].

Theorem 1 Let X, Y, Z be Banach spaces, $x_0 \in X, y_0 \in Y$ and $r > 0$. Assume that $F : B_r(x_0) \times B_r(y_0) \rightarrow Z$ satisfies the following conditions

1. $F(x_0, y_0) = 0$,
2. F is continuous at (x_0, y_0) ,
3. F is differentiable at (x_0, y_0) and $F'_y(x_0, y_0)$ has a bounded inverse,
4. $\|F(x, y) - F(x, y_0) - (F(x_0, y) - F(x_0, y_0))\| \leq L_x \|y - y_0\|$, for any $x \in B_r(x_0)$, $y \in B_r(y_0)$, where $L_x \rightarrow 0$ as $x \rightarrow x_0$.

Then there exists $\alpha > 0$ and $\beta > 0$ such that for any $x \in B_\alpha(x_0)$ the equation

$$F(x, y) = 0 \quad (2)$$

has a unique solution $y = f(x)$ in $B_\beta(y_0)$.

Proof. Let $A_x : B_r(y_0) \rightarrow B_r(y_0)$ be defined as follows $A_x(y) = y - [F'_y(x_0, y_0)]^{-1} F(x, y)$. Clearly, the equation

$$A_x(y) = y \quad (3)$$

is equivalent to (2).

To prove the existence of solutions to (3) we apply the contracting mappings principle. To this end we show that for any $\beta > 0$ sufficiently small there exists $\alpha > 0$ such that for $x \in B_\alpha(x_0)$ the mapping A_x contracts and it maps the ball $B_\beta(y_0)$ into itself. First, using assumptions 4 we evaluate $A_x(y) - A_x(y_0)$ as follows

$$\begin{aligned} \|A_x(y) - A_x(y_0)\| &= \left\| y - y_0 - [F'_y(x_0, y_0)]^{-1} (F(x_0, y) - F(x_0, y_0)) + \right. \\ &\quad \left. + [F'_y(x_0, y_0)]^{-1} (F(x_0, y) - F(x_0, y_0) - (F(x, y) - F(x, y_0))) \right\| \leq \\ &\leq \left\| [F'_y(x_0, y_0)]^{-1} (F(x_0, y) - F(x_0, y_0) - F'_y(x_0, y_0)(y - y_0)) \right\| + \\ &\quad + L_x \left\| [F'_y(x_0, y_0)]^{-1} \right\| \cdot \|y - y_0\|. \end{aligned}$$

Thus, since F is differentiable at (x_0, y_0) and $L_x \rightarrow 0$ as $x \rightarrow x_0$ then for a fixed $\beta > 0$ the constant $\lambda > 0$ can be chosen sufficiently small so that

$$\|A_x(y) - A_x(y_0)\| \leq q \|y - y_0\|, \quad \text{for some } q < 1 \text{ and any } x \in B_\alpha(x_0), y \in B_\beta(y_0).$$

Let us now evaluate $\|A_x(y_0) - y_0\|$. We have

$$\begin{aligned} \|A_x(y_0) - y_0\| &\leq \left\| [F'_y(x_0, y_0)]^{-1} \right\| \cdot \|F(x, y_0)\| = \\ &= \left\| [F'_y(x_0, y_0)]^{-1} \right\| \cdot \|F(x, y_0) - F(x_0, y_0)\|. \end{aligned}$$

Therefore, we can diminish $\alpha > 0$ in such a way that

$$\|A_x(y_0) - y_0\| \leq \beta(1 - q), \quad \text{for any } x \in B_\alpha(x_0).$$

Combining the estimation obtained we arrive to

$$\begin{aligned}\|A_x(y) - y_0\| &\leq \|A_x(y) - A_x(y_0)\| + \|A_x(y_0) - y_0\| \leq \\ &\leq q\|y - y_0\| + \beta(1 - q) \leq q\beta + \beta(1 - q) = \beta.\end{aligned}$$

Thus, for any $x \in B_\alpha(x_0)$ the map A_x maps the closed ball $\overline{B}_\beta(y_0)$ into itself and it contracts in this ball. Therefore, for any $x \in B_\alpha(x_0)$ the map A_x has a unique fixed point $y = f(x)$ in this ball, that implies

$$f(x) = f(x) - [F'_y(x_0, y_0)]^{-1} F(x, f(x))$$

or, equivalently, $F(x, f(x)) = 0$. □

Next theorem is the main result of the paper. It can be derived also from the Baitukenov's theorem [1]. But a proof of [1] did not appear in the literature and, thus, we found reasonable to give a proof independent of the Baitukenov's theorem.

Theorem 2 *Let T, V, E, Z be Banach spaces and $t_0 \in T$, $v_0 \in V$ and $\varepsilon_0 \in E$. Assume that $F : B_r(t_0) \times B_r(v_0) \times B_r(\varepsilon_0) \rightarrow Z$ satisfies the following assumptions*

(i) $F(t_0, v_0, \varepsilon_0) = 0$,

(ii) F is continuous at $(t_0, v_0, \varepsilon_0)$,

(iii) $F'_t(t_0, v_0, \varepsilon_0)$ has a bounded inverse,

(iv) there exists $L_{\varepsilon, v} \rightarrow 0$ as $(\varepsilon, v) \rightarrow (\varepsilon_0, v_0)$ such that

$$\|F(t_1, v, \varepsilon) - F(t_2, v, \varepsilon) - F(t_1, v_0, \varepsilon_0) + F(t_2, v_0, \varepsilon_0)\| \leq L_{\varepsilon, v} \|t_1 - t_2\|$$

for any $t_1, t_2 \in B_r(t_0)$, $v \in B_r(v_0)$, $\varepsilon \in B_r(\varepsilon_0)$.

(v) there exists $K > 0$ and $L_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$ such that

$$\|F(t_1, v_1, \varepsilon) - F(t_1, v_1, \varepsilon) - F(t_2, v_2, \varepsilon_0) + F(t_2, v_1, \varepsilon_0)\| \leq (L_\varepsilon + K\|t_1 - t_2\|) \cdot \|v_1 - v_2\|$$

for any $t_1, t_2 \in B_r(t_0)$, $v \in B_r(v_0)$, $\varepsilon \in B_r(\varepsilon_0)$.

(vi) $(t, v) \rightarrow F(t, v, \varepsilon_0)$ is continuously differentiable in $B_r(t_0) \times B_r(v_0)$,

(vii) F is Lipschitz in $B_r(t_0) \times B_r(v_0) \times B_r(\varepsilon_0)$,

(viii) *The Banach space T possesses the following property: for any $t \in T$ there exists an element $\{t\}$ of T such that $\{t\}^*t = \|t\|$. Moreover $\{t\}$ is uniformly bounded whenever t varies in a bounded set.*

Then there exists $\alpha > 0$ and $\beta > 0$ such that for any $(v, \varepsilon) \in B_\alpha(v_0, \varepsilon_0)$ the equation

$$F(t, v, \varepsilon) = 0 \quad (4)$$

has a unique solution $t = \theta(v, \varepsilon)$ in $B_\beta(t_0)$. Moreover, for any $\Delta > 0$ there exists $\delta > 0$ such that

$$\|\theta(v_1, \varepsilon) - \theta(v_2, \varepsilon)\| \leq \left(\left\| [F'_t(t_0, v_0, \varepsilon_0)]^{-1} F'_v(t_0, v_0, \varepsilon_0) \right\| + \Delta \right) \|v_1 - v_2\| \quad (5)$$

for any $v_1, v_2 \in B_\delta(v_0)$, $\varepsilon \in B_\delta(\varepsilon_0)$.

Proof. Assumptions (i), (ii), (iii) and (iv) imply assumptions 1, 2, 3 and 4 of theorem 1 with $X = T \times V$, $x = (v, \varepsilon)$. Therefore, the conclusion about the existence of $t = \theta(v, \varepsilon)$ solving (4) follows from lemma 1 and it remains to prove (5).

Let $\Delta > 0$. From the classical implicit function theorem and assumption (vi) we have that there exists $\delta > 0$ such that

$$\|\theta(v_1, \varepsilon_0) - \theta(v_2, \varepsilon_0)\| \leq \left(-[F'_t(t_0, v_0, \varepsilon_0)]^{-1} F'_v(t_0, v_0, \varepsilon_0) + \frac{\Delta}{2} \right) \|v_1 - v_2\|$$

for any $v_1, v_2 \in B_\delta(v_0)$. Therefore, to prove (5) it is enough to show that $\delta > 0$ can be diminished in such a way that

$$\|\theta(v_1, \varepsilon) - \theta(v_2, \varepsilon) - (\theta(v_1, \varepsilon_0) - \theta(v_2, \varepsilon_0))\| \leq \frac{\Delta}{2} \|v_1 - v_2\|$$

for any $v_1, v_2 \in B_\delta(v_0)$, $\varepsilon \in B_\delta(\varepsilon_0)$.

Let $\eta > 0$ be fixed. Then by (vi) there exists $d > 0$ such that

$$F'_t(t_2, v_0, \varepsilon_0)(t_1 - t_2) = F(t_1, v_0, \varepsilon_0) - F(t_2, v_0, \varepsilon_0) + \tilde{\gamma}(t_1, t_2) \cdot \|t_1 - t_2\|,$$

where

$$\|\tilde{\gamma}(t_1, t_2)\| \leq \eta \quad \text{for any } t_1, t_2 \in B_d(t_0). \quad (6)$$

Without loss of generality we can assume that

$$0 < d < \eta. \quad (7)$$

For an auxiliary $v \in B_\delta(v_0)$ we consider

$$\begin{aligned} t_1 - t_2 &= [F'_t(t_2, v_0, \varepsilon_0)]^{-1} F'_t(t_2, v_0, \varepsilon_0)(t_1 - t_2) = \\ &= [F'_t(t_2, v_0, \varepsilon_0)]^{-1} (F(t_1, v_0, \varepsilon_0) - F(t_2, v_0, \varepsilon_0) + \tilde{\gamma}(t_1, t_2) \cdot \|t_1 - t_2\|) = \\ &= [F'_t(t_2, v_0, \varepsilon_0)]^{-1} (F(t_1, v, \varepsilon) - F(t_2, v, \varepsilon) + \tilde{\gamma}(t_1, t_2) \cdot \|t_1 - t_2\| + \\ &\quad + F(t_1, v_0, \varepsilon_0) - F(t_2, v_0, \varepsilon_0) - (F(t_1, v, \varepsilon) - F(t_2, v, \varepsilon))). \end{aligned}$$

By (iv) we can diminish $\delta > 0$ in such a way that

$$\|F(t_1, v_0, \varepsilon_0) - F(t_2, v_0, \varepsilon_0) - (F(t_1, v, \varepsilon) - F(t_2, v, \varepsilon))\| \leq \eta \|t_1 - t_2\|$$

for any $t_1, t_2 \in B_d(t_0)$, $v \in B_\delta(v_0)$ and $\varepsilon \in B_\delta(\varepsilon_0)$. Therefore, taking into account (6) we have that

$$t_1 - t_2 = [F'_t(t_2, v_0, \varepsilon_0)]^{-1} (F(t_1, v, \varepsilon) - F(t_2, v, \varepsilon) - \widehat{\gamma}(t_1, t_2, v) \|t_1 - t_2\|), \quad (8)$$

where $\|\widehat{\gamma}(t_1, t_2, v)\| \leq 2\eta$, for any $t_1, t_2 \in B_d(t_0)$, $v \in B_\delta(v_0)$, $\varepsilon \in B_\delta(\varepsilon_0)$.

Since $F(\theta(v_2, \varepsilon), v_2, \varepsilon) = 0 = F(\theta(v_1, \varepsilon), v_1, \varepsilon)$ taking $t_1 = \theta(v_1, \varepsilon)$, $t_2 = \theta(v_2, \varepsilon)$, $v = v_2$ we obtain from (8) that

$$\begin{aligned} \theta(v_1, \varepsilon) - \theta(v_2, \varepsilon) &= [F'_t(\theta(v_2, \varepsilon), v_0, \varepsilon_0)]^{-1} (F(\theta(v_1, \varepsilon), v_2, \varepsilon) - F(\theta(v_1, \varepsilon), v_1, \varepsilon) + \\ &\quad + \widehat{\gamma}(\theta(v_1, \varepsilon), \theta(v_2, \varepsilon), v_2) \cdot \|\theta(v_1, \varepsilon) - \theta(v_2, \varepsilon)\|) \end{aligned}$$

for any $v_1, v_2 \in B_\delta(v_0)$, $\varepsilon \in B_\delta(\varepsilon_0)$. By assumption (viii) we have

$$\begin{aligned} \theta(v_1, \varepsilon) - \theta(v_2, \varepsilon) &= [F'_t(\theta(v_2, \varepsilon), v_0, \varepsilon_0)]^{-1} (F(\theta(v_1, \varepsilon), v_2, \varepsilon) - F(\theta(v_1, \varepsilon), v_1, \varepsilon)) + \\ &\quad + \widehat{\gamma}(\theta(v_1, \varepsilon), \theta(v_2, \varepsilon), v_2) \left\{ \overline{\theta(v_1, \varepsilon) - \theta(v_2, \varepsilon)} \right\} (\theta(v_1, \varepsilon) - \theta(v_2, \varepsilon)), \end{aligned}$$

where $(v_1, v_2, \varepsilon) \mapsto \left\{ \overline{\theta(v_1, \varepsilon) - \theta(v_2, \varepsilon)} \right\}$ is bounded on $B_\delta(v_0) \times B_\delta(v_0) \times B_\delta(\varepsilon_0)$. Since $\eta > 0$ can be chosen sufficiently small we can consider that $I - \widehat{\gamma}(\theta(v_1, \varepsilon), \theta(v_2, \varepsilon), v_2) \left\{ \overline{\theta(v_1, \varepsilon) - \theta(v_2, \varepsilon)} \right\}$ is invertible for any $v_1, v_2 \in B_\delta(v_0)$, $\varepsilon \in B_\delta(\varepsilon_0)$. Thus, we can rewrite the previous expression as follows

$$\begin{aligned} \theta(v_1, \varepsilon) - \theta(v_2, \varepsilon) &= \left(I - \widehat{\gamma}(\theta(v_1, \varepsilon), \theta(v_2, \varepsilon), v_2) \left\{ \overline{\theta(v_1, \varepsilon) - \theta(v_2, \varepsilon)} \right\} \right)^{-1} \circ \\ &\quad \circ [F'_t(\theta(v_2, \varepsilon), v_0, \varepsilon_0)]^{-1} (F(\theta(v_1, \varepsilon), v_2, \varepsilon) - F(\theta(v_1, \varepsilon), v_1, \varepsilon)) \end{aligned}$$

for any $v_1, v_2 \in B_\delta(v_0)$, $\varepsilon \in B_\delta(\varepsilon_0)$.

By the conclusion of theorem 1 the function θ is continuous at (v_0, ε_0) and we can diminish $\delta > 0$ also in such a way that

$$\|\theta(v, \varepsilon) - t_0\| \leq \frac{d}{2} \quad \text{for any } v \in B_\delta(v_0), \varepsilon \in B_\delta(\varepsilon_0). \quad (9)$$

Combining (7) and (9) we have

$$\|\theta(v, \varepsilon) - \theta(v, \varepsilon_0)\| \leq \eta, \quad \text{for any } v \in B_\delta(v_0), \varepsilon \in B_\delta(\varepsilon_0). \quad (10)$$

Thus, using assumption (vii) we have the following expression for $\theta(v_1, \varepsilon) - \theta(v_2, \varepsilon)$

$$\theta(v_1, \varepsilon) - \theta(v_2, \varepsilon) = [F'_t(\theta(v_2, \varepsilon), v_0, \varepsilon_0)]^{-1} (F(\theta(v_1, \varepsilon), v_2, \varepsilon) - F(\theta(v_1, \varepsilon), v_1, \varepsilon)) + \gamma(v_1, v_2, \varepsilon), \quad (11)$$

where

$$\|\gamma(v_1, v_2, \varepsilon)\| \leq L_\eta \|v_1 - v_2\|, \quad \text{for any } v_1, v_2 \in B_\delta(v_0), \varepsilon \in B_\delta(\varepsilon_0) \text{ and } L_\eta \rightarrow 0 \text{ as } \eta \rightarrow 0$$

(of course, $\delta > 0$ depends on $\eta > 0$ as well).

Formula (11) allows us to evaluate $\|\theta(v_1, \varepsilon) - \theta(v_2, \varepsilon) - \theta(v_1, \varepsilon_0) + \theta(v_2, \varepsilon_0)\|$ as follows

$$\begin{aligned}
& \|\theta(v_1, \varepsilon) - \theta(v_2, \varepsilon) - \theta(v_1, \varepsilon_0) + \theta(v_2, \varepsilon_0)\| \leq \\
& \leq \left\| [F'_t(\theta(v_2, \varepsilon_0), v_0, \varepsilon_0)]^{-1} \right\| \cdot \\
& \quad \cdot \|F(\theta(v_1, \varepsilon), v_2, \varepsilon) - F(\theta(v_1, \varepsilon), v_1, \varepsilon) - F(\theta(v_1, \varepsilon_0), v_2, \varepsilon_0) + F(\theta(v_1, \varepsilon_0), v_1, \varepsilon_0)\| + \\
& + \left\| [F'_t(\theta(v_2, \varepsilon), v_0, \varepsilon_0)]^{-1} - [F'_t(\theta(v_2, \varepsilon_0), v_0, \varepsilon_0)]^{-1} \right\| \cdot \\
& \quad \cdot \|F(\theta(v_1, \varepsilon), v_2, \varepsilon) - F(\theta(v_1, \varepsilon), v_1, \varepsilon)\| + \\
& + \|\gamma(v_1, v_2, \varepsilon)\| + \|\gamma(v_1, v_2, \varepsilon_0)\|,
\end{aligned}$$

for any $v_1, v_2 \in B_\delta(v_0)$, $\varepsilon \in B_\delta(\varepsilon_0)$. From (10) we can conclude that

$$\left\| [F'_t(\theta(v_2, \varepsilon), v_0, \varepsilon_0)]^{-1} - [F'_t(\theta(v_2, \varepsilon_0), v_0, \varepsilon_0)]^{-1} \right\| \leq K_\eta$$

for any $v \in B_\delta(v_0)$, $\varepsilon \in B_\delta(\varepsilon_0)$, where $K_\eta \rightarrow 0$ as $\eta \rightarrow 0$. Then, fixing some $K_1 > 0$ such that $[F'_t(\theta(v_2, \varepsilon_0), v_0, \varepsilon_0)]^{-1} \leq K_1$ for any $v_2 \in B_\delta(v_0)$ and applying assumptions (v) and (vii) we have

$$\begin{aligned}
& \|\theta(v_1, \varepsilon) - \theta(v_2, \varepsilon) - \theta(v_1, \varepsilon_0) + \theta(v_2, \varepsilon_0)\| \leq \\
& \leq K_1(L_\varepsilon + K\|\theta(v_1, \varepsilon) - \theta(v_1, \varepsilon_0)\|) \cdot \|v_1 - v_2\| + K_\eta L\|v_1 - v_2\| + 2L_\eta\|v_1 - v_2\| \leq \\
& \leq (K_1L_\varepsilon + K_1K\eta + K_\eta L + 2L_\eta) \cdot \|v_1 - v_2\|
\end{aligned}$$

for any $v_1, v_2 \in B_\delta(v_0)$, $\varepsilon \in B_\delta(\varepsilon_0)$.

Therefore, given $\Delta > 0$ we can find $\eta > 0$ and $\delta > 0$ (which depends on $\eta > 0$) such that

$$\|\theta(v_1, \varepsilon) - \theta(v_2, \varepsilon) - (\theta(v_1, \varepsilon_0) - \theta(v_2, \varepsilon_0))\| \leq \frac{\Delta}{2}\|v_1 - v_2\|$$

for any $v_1, v_2 \in B_\delta(v_0)$, $\varepsilon \in B_\delta(\varepsilon_0)$. Thus the proof is complete. □

3. An application. Consider the second order differential equation

$$\ddot{u} + u = -\varepsilon \text{sign}(u) + \varepsilon g(t, u, \dot{u}), \tag{12}$$

where g is continuously differentiable and 2π -periodic in time. The change of variables

$$\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

allows us to transform (12) into the following system

$$\begin{aligned}
\dot{x}_1 &= \varepsilon \sin(t) \text{sign}(x_1 \cos t + x_2 \sin t) \\
\dot{x}_2 &= -\varepsilon \cos(t) \text{sign}(x_1 \cos t + x_2 \sin t) + \varepsilon \cdot \boxed{\text{continuously differentiable terms}}.
\end{aligned} \tag{13}$$

We assume that for any $v \in \mathbb{R}^2$ system (13) has an unique absolutely continuous solution $x(\cdot, v, \varepsilon)$ defined on $[0, T]$ and such that

(F) $x(t, v, \varepsilon)$ possesses the following representation $x(t, v, \varepsilon) = v + \varepsilon y(t, v, \varepsilon)$, where $(t, v) \mapsto y(t, v, \varepsilon)$ is locally Lipschitz uniformly with respect to small $\varepsilon > 0$.

Remark 1 A natural example when all the imposed assumptions are satisfied is when (12) models a pendulum with dry friction (see [5, example 2.2.3]).

Consider

$$F(t, v, \varepsilon) = x_1(t, v, \varepsilon) \cos t + x_2(t, v, \varepsilon) \sin t.$$

The following proposition is crucial when generalizing the second Bogolyubov's theorem [2] for discontinuous systems of form (13).

Proposition 1 Assume that (F) is satisfied. Let $t_0 \in (a, b) \subset (0, 2\pi)$ be the only zero of $F(\cdot, v_0, 0)$ on $[a, b]$ and define

$$R = \frac{1}{|-[v_0]_1 \sin t_0 + [v_0]_2 \cos t_0|}.$$

Then there exists a function $\theta : B_\delta(v_0) \times B_\delta(v_0) \rightarrow [a, b]$ such that given $\Delta > 0$ there exists $\delta > 0$ such that $(t, v) \mapsto F(t, v, \varepsilon)$ does not vanish on

$$([a, b] \setminus [\theta(v_1, \varepsilon) - (R + \Delta)\|v_1 - v_2\|, \theta(v_1, \varepsilon) + (R + \Delta)\|v_1 - v_2\|]) \times [v_1, v_2] \quad (14)$$

whenever $v_1, v_2 \in B_\delta(v_0)$, $\varepsilon \in (0, \delta)$.

Proof. Let us show that assumptions of theorem 2 are satisfied with $T = \mathbb{R}$, $V = \mathbb{R}^2$, $E = \mathbb{R}$, $Z = \mathbb{R}^2$, $\varepsilon_0 = 0$. Properties (i) and (ii) are straightforward and we, therefore, start with (iii).

(iii) Since $x'_t(t, v, 0) \equiv 0$ then $F'_t(t_0, v_0, 0) = -[v_0]_1 \sin t_0 + [v_0]_2 \cos t_0$ which, as it can be easily verified, equals to 0 if and only if $F(t_0, v_0, 0) \neq 0$.

(iv) The conclusion follows observing that $x(t_1, v, \varepsilon) - x(t_2, v, \varepsilon) - x(t_1, v_0, 0) - x(t_2, v_0, 0) = \varepsilon(y(t_1, v, \varepsilon) - y(t_2, v, \varepsilon))$,

(v) Follows from the obvious identity $x(t_1, v_2, \varepsilon) - x(t_1, v_1, \varepsilon) - x(t_2, v_2, 0) + x(t_2, v_1, 0) = \varepsilon(y(t_1, v_2, \varepsilon) - y(t_1, v_1, \varepsilon))$.

(vi) $F(t, v, 0) = v_1 \cos t + v_2 \sin t$ and so is continuously differentiable in v and t .

(vii) Follows from assumption (F).

(viii) The property holds true with $\{t\} = \text{sign}(t)$.

Therefore, theorem 2 applies and the function $t = \theta(v, \varepsilon)$ solving (4) and satisfying (5) exists. Moreover,

$$\left\| [F'_t(t_0, v_0, 0)]^{-1} F'_v(t_0, v_0, 0) \right\| = \left\| \frac{1}{-[v_0]_1 \sin t_0 + [v_0]_2 \cos t_0} (\cos t_0, \sin t_0) \right\| = R.$$

To prove (14) we recover that theorem 2 claims that for any $v \in B_\delta(v_0)$, $\varepsilon \in (0, \delta)$ the function $F(\cdot, v, \varepsilon)$ has a unique zero in $B_\delta(t_0)$ which is $\theta(v, \varepsilon)$. On the other hand since $t_0 \in (a, b)$ is the only zero of $F(\cdot, v_0, \varepsilon_0)$ then $\delta > 0$ can be diminished, if necessary, in such a way that $F(\cdot, v, \varepsilon)$ does not vanish on $[a, b] \setminus \{\theta(v, \varepsilon)\}$ for any $v \in B_\delta(v_0)$, $\varepsilon \in (0, \delta)$. Fix some $v_1, v_2 \in B_\delta(v_0)$. From conclusion (5) of theorem 2 we have that $\|\theta(v_1, \varepsilon) - \theta(v_2, \varepsilon)\| \leq (R + \Delta)\|v_1 - v_2\|$ for any $v \in [v_1, v_2]$, $\varepsilon \in (0, \delta)$, which implies that

$$[a, b] \setminus \{\theta(v, \varepsilon)\} \supset [a, b] \setminus [\theta(v_1, \varepsilon) - (R + \Delta)\|v_1 - v_2\|, \theta(v_1, \varepsilon) + (R + \Delta)\|v_1 - v_2\|]$$

for any $v \in [v_1, v_2]$, $\varepsilon \in (0, \delta)$. This finishes the proof. □

Remark 2 *The conclusion of proposition 1 can not be achieved with the classical implicit function theorem since nothing can be said around the continuity of the derivative of $(t, v, \varepsilon) \mapsto F(t, v, \varepsilon)$ with respect to t unless $\varepsilon = 0$.*

References

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